Categories: How I Learned to Stop Worrying and Love Two Sorts

Alessandra Palmigiano
joint work with
W. Conradie, S. Frittella, A. Tzimoulis and N. Wijnberg.

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Introduction

Problem: Understanding relational semantics for lattice-based (e.g. substructural) logics:

- Not 1 set of possible worlds, but 2 domains: states and **co-states** (???)
- Semantic interpretation of $\Box$ and $\Diamond$ very different from the usual clauses in Boolean and distributive settings

$$\mathbb{M}, x \succ \Diamond\phi \quad \text{iff} \quad \textbf{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \phi, \text{ then } xRa$$
$$\mathbb{M}, a \Vdash \Diamond\phi \quad \text{iff} \quad \textbf{for all } x \in X, \text{ if } \mathbb{M}, x \succ \Diamond\phi, \text{ then } a \perp x$$

- Satisfaction $\Vdash$ and co-satisfaction $\succ$ (???)

Proposed solution: Formulas as categories (as understood in business science).

Advantage: Surprisingly natural epistemic interpretation!

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Two-sorted semantics for lattice-based modal logics

Polarity. $\mathbb{P} = (A, X, \perp)$ with A and X sets and $\perp \subseteq A \times X$.

Galois connection. $(\cdot)^{\uparrow} : \mathcal{P}A \to \mathcal{P}X$ and $(\cdot)^{\downarrow} : \mathcal{P}X \to \mathcal{P}A$ s.t. for all $B \subseteq A$ and $Y \subseteq X$,

- $B^{\uparrow} := \{x \in X \mid \forall a(a \in B \to a \perp x)\}$,
- $Y^{\downarrow} := \{a \in A \mid \forall x(x \in Y \to a \perp x)\}$.

Closed sets. $B = B^{\uparrow\downarrow}$ and $Y = Y^{\downarrow\uparrow}$.

Lattice of closed sets. Let $C(A)$ (resp. $C(X)$) be the closed subsets of A (resp. X).

$$\mathbb{P}^{+} = (C(A), \bigcap, \bigvee, \varnothing^{\uparrow\downarrow}, A) \cong^{\partial} (C(X), \bigcap, \bigvee, \varnothing^{\downarrow\uparrow}, X).$$

Concept lattice of $\mathbb{P}$. Lattice of tuples (B, Y) s.t.

$$Y = B^{\uparrow} \quad \text{and} \quad B = Y^{\downarrow}.$$

RS-polarities

RS-Polarity. $(A, X, \perp)$ such that

(S1) $b \neq c$ implies $\perp[b] \neq \perp[c]$, for all $b, c \in A$

(S2) $y \neq z$ implies $\perp^{-1}[y] \neq \perp^{-1}[z]$, for all $y, z \in X$

(R1) $a^{\uparrow\downarrow}$ completely join-irreducible in $\mathbb{P}^+$ for every $a \in A$

(R2) $x^{\downarrow\uparrow}$ completely meet-irreducible in $\mathbb{P}^+$ for every $x \in X$

Lattice of closed sets.
$\mathbb{P}^+ = (C(A), \bigcap, \bigvee, \varnothing, A) \cong^{\partial} (C(X), \bigcap, \bigvee, \varnothing, X)$.

Perfect lattices and RS-polarities. For any perfect lattice $\mathbb{L}$, $\mathbb{L}_+ := (J^\infty(\mathbb{L}), M^\infty(\mathbb{L}), \leq)$ is an RS-polarity.

Theorem. For any RS-polarity $\mathbb{P}$ and perfect lattice $\mathbb{L}$,

$$(\mathbb{P}^+)_+ \cong \mathbb{P} \quad \text{and} \quad \mathbb{L} \cong (\mathbb{L}_+)^+$$

RS-frames

Language. $\mathcal{L} = 0 \mid 1 \mid \vee \mid \wedge \mid \Box$

RS-frame. $\mathbb{F} = (\mathbb{P}, R)$ such that

- $\mathbb{P} = (A, X, \perp)$ is an RS-polarity
- $R \subseteq A \times X$
- $R[b]$ and $R^{-1}[y]$ are closed sets, for all $b \in A$ and $y \in X$.

RS-models. $\mathbb{M} = (\mathbb{F}, V)$ s.t.

- $\mathbb{F}$ RS-frame
- for all $p \in$ **AtProp**,

$$V(p) = (V_1(p),\ V_2(p))$$

with $V_1(p) = V_2(p)^{\downarrow}$ and $V_2(p) = V_1(p)^{\uparrow}$.

Interpretation of lattice-based modal logic on RS-frames

$\mathbb{M}, a \Vdash 0$	never		$\mathbb{M}, x \succ 0$	always	
$\mathbb{M}, a \Vdash 1$	always		$\mathbb{M}, x \succ 1$	never	

$\mathbb{M}, a \Vdash p$	iff	$a \in V_1(p)$	$\mathbb{M}, x \succ p$	iff	$x \in V_2(p)$
$\mathbb{M}, a \Vdash \mathbf{i}$	iff	$a \in V_1(\mathbf{i})$	$\mathbb{M}, x \succ \mathbf{i}$	iff	$x \in V_2(\mathbf{i})$
$\mathbb{M}, a \Vdash \mathbf{m}$	iff	$a \in V_1(\mathbf{m})$	$\mathbb{M}, x \succ \mathbf{m}$	iff	$x \in V_2(\mathbf{m})$

$\mathbb{M}, a \Vdash \phi \wedge \psi$ iff $\mathbb{M}, a \Vdash \phi$ and $\mathbb{M}, a \Vdash \psi$

$\mathbb{M}, x \succ \phi \wedge \psi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \phi \wedge \psi$, then $a \perp x$

$\mathbb{M}, a \Vdash \phi \vee \psi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi \vee \psi$, then $a \perp x$

$\mathbb{M}, x \succ \phi \vee \psi$ iff $\mathbb{M}, x \succ \phi$ and $\mathbb{M}, x \succ \psi$

$\mathbb{M}, a \Vdash \Box\phi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi$, then aRx

$\mathbb{M}, x \succ \Box\phi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box\phi$, then $a \perp x$

Standard translation

$$\mathrm{ST}_a(0) := a \neq a \qquad \mathrm{ST}_a(1) := a = a \qquad \mathrm{ST}_a(p) := P_1(a)$$
$$\mathrm{ST}_x(0) := x = x \qquad \mathrm{ST}_x(1) := x \neq x \qquad \mathrm{ST}_x(p) := P_2(x)$$
$$\mathrm{ST}_a(\mathbf{j}) := \forall x[a \bot x \to j \bot x] \qquad \mathrm{ST}_x(\mathbf{j}) := j \perp x$$
$$\mathrm{ST}_a(\mathbf{m}) := a \perp m \qquad \mathrm{ST}_x(\mathbf{m}) := \forall a[a \bot m \to a \bot x]$$
$$\mathrm{ST}_a(\phi \vee \psi) := \forall x[\mathrm{ST}_x(\phi \vee \psi) \to a \perp x]$$
$$\mathrm{ST}_x(\phi \vee \psi) := \mathrm{ST}_x(\phi) \wedge \mathrm{ST}_x(\psi)$$
$$\mathrm{ST}_a(\phi \wedge \psi) := \mathrm{ST}_a(\phi) \wedge \mathrm{ST}_a(\psi)$$
$$\mathrm{ST}_x(\phi \wedge \psi) := \forall a[\mathrm{ST}_a(\phi \wedge \psi) \to a \perp x]$$
$$\mathrm{ST}_a(\Box\phi) := \forall x[\mathrm{ST}_x(\phi) \to aRx]$$
$$\mathrm{ST}_x(\Box\phi) := \forall a[\mathrm{ST}_a(\Box\phi) \to a \perp x]$$

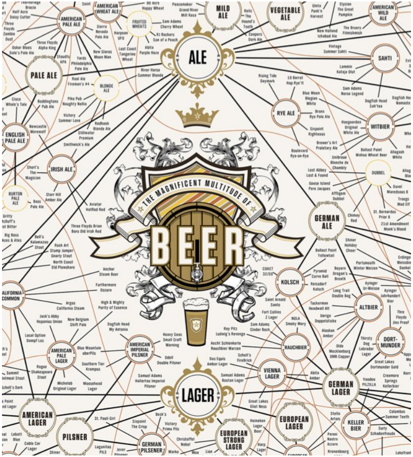
Main property.

$$\mathbb{M} \Vdash \phi \leq \psi \ \text{ iff } \ \mathbb{M} \models \forall a[\mathrm{ST}_a(\phi) \to \mathrm{ST}_a(\psi)]$$

Categorization theory

From Wikipedia:

Categorization is the process in which ideas and objects are recognized, differentiated, and understood.
Ideally, a category illuminates a relationship between the subjects and objects of knowledge.
Categorization is fundamental in language, prediction, inference, decision making and in all kinds of environmental interaction.



Categorization theory and RS-models via Formal Concept Analysis

Let $\mathbb{F} = (\mathbb{P}, R)$ with

- $\mathbb{P} = (A, X, \perp)$ database
- A set of objects (e.g. car models currently on sale)
- X set of features (e.g. electric, 3 doors, red...)
- $\perp$ incidence relation: $a \perp x$ iff object a has feature x
- $R \subseteq A \times X$ knowledge/perception/beliefs of a given agent:
  aRx iff object a has feature x according to the agent
- $a^{\uparrow}$ set of features of object a
- $x^{\downarrow}$ set of objects having feature x
- $B^{\uparrow}$ set of features shared by all objects in B
- $Y^{\downarrow}$ set of objects satisfying all features in Y
- $\mathbb{P}^{+}$ concept lattice arising from database $\mathbb{P}$

Categories as social constructs

Social interaction is key to categorization theory:

- categories arise from factual information about the world.
- **However**, what they mean critically depends on how people **perceive** them and **agree** about them

Three aspects of categorization theory:

- factual truth
- subjective perception / knowledge / beliefs
- social interaction

Epistemic interpretation of $\Box$

In an RS-frame $\mathbb{F} = (\mathbb{P}, R)$:

- $R \subseteq A \times X$ encodes perception of a given agent about objects and their features
- aRx reads 'object a has feature x according to the agent'
- $\Box\phi$ reads 'category which the agent understands as ϕ'

Factivity of knowledge. $\Box\phi \leq \phi$

$$
\begin{array}{ll}
 & \forall p(\Box p \leq p) \\
\text{iff} & \forall \mathbf{m}(\Box \mathbf{m} \leq \mathbf{m}) \\
\text{iff} & \forall a \forall \mathbf{m}[\mathrm{ST}_a(\Box \mathbf{m}) \rightarrow \mathrm{ST}_a(\mathbf{m})] \\
\text{iff} & \forall a \forall m(aRm \rightarrow a \perp m), \\
 & \text{if } a \text{ has } m \text{ according to the agent, then } a \text{ has } m \text{ in reality}
\end{array}
$$

Epistemic interpretation of $\Box$, continued

Positive introspection. $\Box\phi \leq \Box\Box\phi$

$$
\begin{array}{ll}
 & \forall p(\Box p \leq \Box\Box p) \\
\text{iff} & \forall \mathbf{m}(\Box \mathbf{m} \leq \Box\Box \mathbf{m}) \\
\text{iff} & \forall a \forall \mathbf{m}(\mathrm{ST}_a(\Box \mathbf{m}) \rightarrow \mathrm{ST}_a(\Box\Box \mathbf{m})) \\
\text{iff} & \forall a \forall m(aRm \rightarrow R^{-1}[m]^{\uparrow} \subseteq R[a]),
\end{array}
$$

Remark. Recall that for every $(B, Y) \in \mathbb{P}^+$,

$$\forall a \forall y[(a \in B \ \& \ y \in Y) \rightarrow a \bot y].$$

Meaning of the latter statement.
By definition, $\Box \mathbf{m} := (R^{-1}[m], R^{-1}[m]^{\uparrow}) \in \mathbb{P}^+$

$$\forall a \forall y[(a \in R^{-1}[m] \ \& \ y \in R^{-1}[m]^{\uparrow}) \rightarrow aRy].$$

Social interaction

RS-frame. Two agents $(\mathbb{P}, R_1, R_2)$.

Axioms. $\Box_i p \leq p$ and $\Box_i p \leq \Box_i \Box_i p$ for $1 \leq i \leq 2$.

Common knowledge type operator. For every $u \in \mathbb{P}^+$, let

$$C(u) := \bigwedge_{s \in S} su,$$

where $s \in S$ is such that $s = (\Box_i \Box_j)^n$ or $s = \Box_i (\Box_j \Box_i)^n$ with $i, j \in \{1, 2\}$, $i \neq j$ and $n \in \mathbb{N}$.

Lemma. C is a fixed point operator.

Intuition. Fixed points of C are understood as socially constructed categories, the membership and description of which are socially agreed upon.

Conclusions & further research

Conclusions

- Categorization theory as a conceptual framework of reference to make sense of RS-semantics for lattice-based modal logic
- Epistemic interpretation capturing knowledge **directly**, rather than by way of uncertainty
- Factivity, positive introspection **retain** their familiar meaning
- Mathematical framework to study categorization phenomena in social science

Further research

- lattice-based fixed point logics as an epistemic framework for categories socially agreed upon (default categories)
- Formalizing dynamics of categories
- Logical environment for analyzing decision-making procedures